Reciprocity principle applicable to reflected radiance measurements and the searchlight problem

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In a recent paper by Di Girolamo *et al.* [J. Geophys. Res. D **103,** 8795 (1998)] a heuristic argument was used to derive a reciprocity principle applicable to reflected solar radiation measurements. Here a formal derivation of this reciprocity principle is presented. It is also demonstrated that a purely spatial reciprocal relationship exists between one-dimensional radiative transfer theory and the three-dimensional searchlight problem for horizontally homogeneous media. © 1999 Optical Society of America

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1. Introduction

In a recent paper by Di Girolamo *et al.*,[1] a form of the reciprocity principle for natural solar radiation (i.e., unpolarized sunlight) was presented as proposition 1:

> "Proposition 1: Let A and B represent two surfaces in space, and let the medium that fills the space be fixed in time during the reciprocal measurements. The general principle of reciprocity may be stated in the following way: the radiance emerging from B in direction Θ_B, caused by illuminating A alone from direction Θ_A with total intensity F, is equal to the radiance emerging from A in direction Θ_A, caused by illuminating B alone from direction Θ_B with total intensity F. The same holds for incident radiances and observed total intensities."

In this statement, the total intensity is the radiance integrated over area. The term general reciprocity is used to distinguish it from the more commonly known directional reciprocity, which involves only the directional and not the spatial attributes of illumination and measurement. Directional reciprocity is valid only for horizontally homogeneous media with constant illumination at all points[2] and for a few other special cases.[1] In contrast, general reciprocity is valid regardless of the medium's heterogeneity, as long as the scattering phase function has time-reflection symmetry everywhere in the medium.

Di Girolamo *et al.*[1] derived proposition 1 by using a heuristic approach for physical insight and pointed to Case[3] for a rigorous mathematical formulation. Case's formulation results in a reciprocity identity from which all other reciprocal relationships, including proposition 1, fall out as special cases. Since the publication of Ref. 1, this author has received several requests to formulate proposition 1 from Case's reciprocity identity, given that its formulation is explicitly derived neither in Ref. 3 nor elsewhere in the literature. This paper, therefore, provides that derivation. A discussion follows the derivation, where a link is made between one-dimensional (1-D) radiative transfer theory and the three-dimensional searchlight problem for horizontally homogeneous media.

2. Derivation

Consider an arbitrarily shaped enclosure with a surface denoted by S, bounding a volume V, as depicted in Fig. 1. The medium contained in V can be an absorbing, emitting, and scattering heterogeneous medium. Let $\mathbf{r}$ denote the position vector, Ω be a unit vector describing direction, and $\mathbf{n}$ be a unit vector that is outward normal to surface S (i.e., $\mathbf{n} \cdot \Omega < 0$ represents a direction incident upon S from outside V). The radiative transfer equation can then be written as

$$\Omega \cdot \nabla I(\mathbf{r}, \Omega) = -\sigma_e(\mathbf{r}) I(\mathbf{r}, \Omega) + \frac{\sigma_s(\mathbf{r})}{4\pi} \int_{4\pi} P(\mathbf{r}, \Omega, \Omega') \times I(\mathbf{r}, \Omega') \mathrm{d}\Omega' + q(\mathbf{r}, \Omega), \qquad (1)$$

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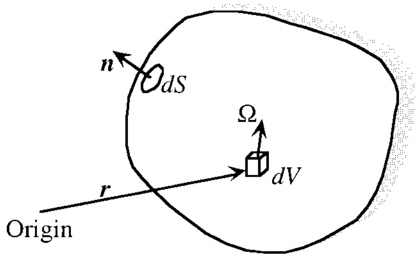

Fig. 1. Arbitrarily shaped enclosure with surface S, bounding a volume V, used to describe the geometry of the derivation.

where I is the radiance, σ_e and σ_s are the volume extinction and scattering coefficients, respectively, P is the scattering phase function normalized as $\int_{4\pi} P(\mathbf{r}, \Omega, \Omega')d\Omega' = 4\pi$, and q is the emission term, which can be represented as the product of the volume absorption coefficient and the Planck function under thermodynamic equilibrium. Let us assume that the scattering phase function has time-reversal symmetry; that is, that

$$P(\mathbf{r}, \Omega, \Omega') = P(\mathbf{r}, -\Omega', -\Omega), \tag{2}$$

which is often true for scatterers encountered in the atmosphere. Based on this assumption, Case[3] derived the following expression:

$$\int_S \int_{\Omega\cdot\mathbf{n}<0} |\Omega \cdot \mathbf{n}|[I_1(\mathbf{r}, -\Omega)I_2^{\mathrm{inc}}(\mathbf{r}, \Omega) - I_1^{\mathrm{inc}}(\mathbf{r}, \Omega) \times I_2(\mathbf{r}, -\Omega)]d\Omega d\mathbf{r} = \int_V \int_{4\pi} [I_2(\mathbf{r}, -\Omega)q_1(\mathbf{r}, \Omega) - I_1(\mathbf{r}, \Omega)q_2(\mathbf{r}, -\Omega)]d\Omega d\mathbf{r}, \tag{3}$$

where $I_1(\mathbf{r}, \Omega)$ is the solution to Eq. (1) within V, bounded by S, subject to the following boundary condition:

$$I_1(\mathbf{r}, \Omega) = I_1^{\mathrm{inc}}(\mathbf{r}, \Omega), \qquad \mathbf{r} \in S, \quad \mathbf{n} \cdot \Omega < 0. \tag{4}$$

Similarly, $I_2(\mathbf{r}, -\Omega)$ is the solution to

$$-\Omega \cdot \nabla I_2(\mathbf{r}, -\Omega) = -\sigma_e(r)I_2(\mathbf{r}, -\Omega) + \frac{\sigma_s(\mathbf{r})}{4\pi} \times \int_{4\pi} P(\mathbf{r}, -\Omega, -\Omega')I_2(\mathbf{r}, -\Omega')d\Omega' + q_2(\mathbf{r}, -\Omega) \tag{5}$$

within V, bounded by S, subject to the following boundary condition:

$$I_2(\mathbf{r}, -\Omega) = I_2^{\mathrm{inc}}(\mathbf{r}, -\Omega), \qquad \mathbf{r} \in S, \quad -\mathbf{n} \cdot \Omega < 0. \tag{6}$$

Equation (3) is the reciprocity identity and is equivalent to Eq. (20) of Ref. 3. Many so-called reciprocity principles can be derived from this identity. If the illumination and the measurement are made in different media, the relative refractive index between the two media needs to be taken into account.[4,5] In the context of proposition 1, the media (space) are the same and there are no emission terms, so Eq. (3) simply becomes

$$\int_S \int_{\Omega\cdot\mathbf{n}<0} \Omega \cdot \mathbf{n}I_1(\mathbf{r}, -\Omega)I_2^{\mathrm{inc}}(\mathbf{r}, \Omega)d\Omega d\mathbf{r} = \int_S \int_{\Omega\cdot\mathbf{n}<0} \Omega \cdot \mathbf{n}I_1^{\mathrm{inc}}(\mathbf{r}, \Omega)I_2(\mathbf{r}, -\Omega)d\Omega d\mathbf{r}. \tag{7}$$

To derive proposition 1 from Eq. (7), let

$$I_1(\mathbf{r}, \Omega) = I(\mathbf{r}, \Omega; A, \Omega_1), \qquad \mathbf{r}, A \in S \tag{8}$$

be the radiance at $\mathbf{r} \in S$ in direction Ω caused by illuminating area $A \in S$ from direction Ω_1, where the illumination is taken to be parallel rays of light from one direction (e.g., sunlight reaching the Earth in the context of Ref. 1) and is given by

$$I_1^{\mathrm{inc}}(\mathbf{r}, \Omega) = \delta(\mathbf{r} - \mathbf{r}_A)\tilde{\delta}(\Omega \cdot \Omega_1)F_1(\mathbf{r}, \Omega), \qquad \mathbf{r}_A \in A, \tag{9}$$

where $\int_S \delta(\mathbf{r} - \mathbf{r}_A)d\mathbf{r} = \text{area } A$, $\delta(\mathbf{r} - \mathbf{r}_A) = 1$ when $\mathbf{r} \in A$, $\delta(\mathbf{r} - \mathbf{r}_A) = 0$ when $\mathbf{r} \notin A$, $\int_{4\pi} \tilde{\delta}(\Omega \cdot \Omega_1)d\Omega = 1$, $\tilde{\delta}(\Omega \cdot \Omega_1) = 1$ when $\Omega \cdot \Omega_1 = 0$, and $\tilde{\delta}(\Omega \cdot \Omega_1) = 0$ when $\Omega \cdot \Omega_1 \neq 0$. Similarly, let

$$I_2(\mathbf{r}, \Omega) = I(\mathbf{r}, \Omega; B, \Omega_2), \qquad \mathbf{r}, B \in S \tag{10}$$

be the radiance at $\mathbf{r} \in S$ in direction Ω caused by illuminating area $B \in S$ from direction Ω_2, where the illumination is given by

$$I_2^{\mathrm{inc}}(\mathbf{r}, \Omega) = \delta(\mathbf{r} - \mathbf{r}_B)\tilde{\delta}(\Omega \cdot \Omega_2)F_2(\mathbf{r}, \Omega), \qquad \mathbf{r}_B \in B. \tag{11}$$

Substituting Eqs. (8)–(11) into Eq. (7) yields

$$\int_S \int_{\Omega\cdot\mathbf{n}<0} \Omega \cdot \mathbf{n}\delta(\mathbf{r} - \mathbf{r}_B)\tilde{\delta}(\Omega \cdot \Omega_2)F_2(\mathbf{r}, \Omega) \times I(\mathbf{r}, -\Omega; A, \Omega_1)d\Omega d\mathbf{r} = \int_S \int_{\Omega\cdot\mathbf{n}<0} \Omega \cdot \mathbf{n}\delta(\mathbf{r} - \mathbf{r}_A) \times \tilde{\delta}(\Omega \cdot \Omega_1)F_1(\mathbf{r}, \Omega)I(\mathbf{r}, -\Omega; B, \Omega_2)d\Omega d\mathbf{r},$$

and carrying out the integration over S and $\mathbf{n} \cdot \Omega < 0$ yields

$$\Omega_2 \cdot \mathbf{n} \int_B F_2(\mathbf{r}, \Omega_2)I(\mathbf{r}, -\Omega_2; A, \Omega_1)d\mathbf{r} = \Omega_1 \cdot \mathbf{n} \int_A F_1(\mathbf{r}, \Omega_1)I(\mathbf{r}, -\Omega_1; B, \Omega_2)d\mathbf{r}. \tag{12}$$

Equation (12) is equivalent to proposition 1 when $F_1 = F_2 = F$. Note that Eq. (12) is more general than proposition 1 inasmuch as it demonstrates that the spatial variability in both the incident and the measured radiation is convoluted. This cannot be deduced from proposition 1. Also, there is no restriction in Eq. (12) that $F_1 = F_2 = F$, as implied in proposition 1.

3. Discussion

In the special case of a horizontally homogeneous medium with constant and complete illumination from one direction, $F_1(\mathbf{r}, \Omega_1)$, $F_2(\mathbf{r}, \Omega_2)$, $I(\mathbf{r}, -\Omega_2; A, \Omega_1)$, and $I(\mathbf{r}, -\Omega_1; B, \Omega_2)$ in Eq. (12) become independent of the spatial component, and we are left with

$$\frac{I(-\Omega_2; \Omega_1)}{\Omega_1 \cdot \mathbf{n} F_1(\Omega_1)} = \frac{I(-\Omega_1; \Omega_2)}{\Omega_2 \cdot \mathbf{n} F_2(\Omega_2)}. \tag{13}$$

This is the more popular form of the reciprocity principle used in optical and geophysical sciences and has been termed the principle of directional reciprocity.[1,6] It is often referred to as the Helmholtz reciprocity principle, even though von Helmholtz's original statement[7] involved radiation at a point (i.e., where areas A and B are infinitesimally small), included the effects of polarization, and was independent of the medium's degree of heterogeneity. Although the effects of polarization have not been included in this paper, arriving at a reciprocity relationship as discussed in Ref. 2 requires symmetry in the phase matrix.

From Eq. (12), a link can be made between 1-D radiative transfer theory and the searchlight problem. In the searchlight problem, the incident radiation is applied to a single point or a finite area from a single direction at the top of a horizontally homogeneous surface, giving way to a three-dimensional radiative transfer solution. In 1-D radiative transfer theory, the entire horizontally homogeneous surface is uniformly illuminated, giving way to a 1-D radiative transfer solution. We can make the link between the two problems by substituting $F_1(\mathbf{r}, \Omega_1) = F_{\mathrm{SL}}(\mathbf{r}, \Omega_1)$, $I(\mathbf{r}, -\Omega_2; A, \Omega_1) = I_{\mathrm{SL}}(\mathbf{r}, -\Omega_2; A, \Omega_1)$, $F_2(\mathbf{r}, \Omega_2) = F_{\mathrm{1D}}(\Omega_2)$, and $I(\mathbf{r}, -\Omega_1; B, \Omega_2) = I_{\mathrm{1D}}(-\Omega_1; \Omega_2)$ into Eq. (12), which yields

$$\Omega_2 \cdot \mathbf{n} \int_{\infty} F_{\mathrm{1D}}(\Omega_2) I_{\mathrm{SL}}(\mathbf{r}, -\Omega_2; A, \Omega_1) \mathrm{d}\mathbf{r} = \Omega_1 \cdot \mathbf{n} \int_A F_{\mathrm{SL}}(\mathbf{r}, \Omega_1) I_{\mathrm{1D}}(-\Omega_1; \Omega_2) \mathrm{d}\mathbf{r}. \tag{14}$$

On the left-hand side of Eq. (14), integration is carried out over the entire illuminated area, which horizontally extends to infinity in 1-D radiative transfer theory. Because the one-dimensional terms are independent of $\mathbf{r}$, Eq. (14) can be rearranged as

$$\frac{\int_{\infty} I_{\mathrm{SL}}(\mathbf{r}, -\Omega_2; A, \Omega_1) \mathrm{d}\mathbf{r}}{\Omega_1 \cdot \mathbf{n} \int_A F_{\mathrm{SL}}(\mathbf{r}, \Omega_1) \mathrm{d}\mathbf{r}} = \frac{I_{\mathrm{1D}}(-\Omega_1; \Omega_2)}{\Omega_2 \cdot \mathbf{n} F_{\mathrm{1D}}(\Omega_2)}. \tag{15}$$

The right-hand side of Eq. (15) is equivalent to the right-hand side of Eq. (13). Equation (15) can therefore be written as

$$\frac{\int_{\infty} I_{\mathrm{SL}}(\mathbf{r}, -\Omega_2; A, \Omega_1) \mathrm{d}\mathbf{r}}{\Omega_1 \cdot \mathbf{n} \int_A F_{\mathrm{SL}}(\mathbf{r}, \Omega_1) \mathrm{d}\mathbf{r}} = \frac{I_{\mathrm{1D}}(-\Omega_2; \Omega_1)}{\Omega_1 \cdot \mathbf{n} F_{\mathrm{1D}}(\Omega_1)}. \tag{16}$$

When the left-hand side of Eq. (16) is defined as the total bidirectional reflectance distribution function, Eq. (16) states that the total bidirectional reflectance distribution function calculated in the searchlight problem is equal to the bidirectional reflectance distribution function calculated in 1-D radiative transfer theory for the same horizontally homogeneous medium. Note that Eq. (16) involves only a switch between the spatial components of illumination and measurement and not in the directional components. Therefore, analogously to Eq. (13), which was termed the principle of directional reciprocity, Eq. (16) may be termed the principle of spatial reciprocity. Like directional reciprocity, spatial reciprocity is valid only for horizontally homogeneous media. Spatial reciprocity can be used, for example, in validating the theory and numerical computations of the searchlight problem. In this case, 1-D radiative transfer theory and numerical calculations provide the benchmark because they are well established, whereas the theory and numerical techniques required for handling the searchlight problem are still active areas of research.[8] Solutions to the searchlight problem are finding utility, for example, in emerging remote-sensing techniques for determining cloud properties.[9]

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